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On a free boundary value problem for the anisotropic N-Laplace operator on an $N-$dimensional ring domain

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Abstract

In this paper we are going to investigate a free boundary value problem for the anisotropic N-Laplace operator on a ring domain $\Omega := \Omega_0 \setminus \overline{\Omega}_1 \subset \mathbb{R}^N$, $N \geq 2$. Our aim is to show that if the problem admits a solution in a suitable weak sense, then the underlying domain Ω is a Wulff shaped ring. The proof makes use of a maximum principle for an appropriate P-function, in the sense of L.E. Payne and some geometric arguments involving the anisotropic mean curvature of the free boundary.

1 Introduction

Let $F : \mathbb{R}^N \to [0, \infty)$, $N \geq 2$, be a norm in $\mathbb{R}^N$, such that

$$\begin{aligned} & F \in C^{3,\alpha}_{loc}\left(\mathbb{R}^N \setminus \{\mathbf{0}\}\right), \text{ with } \alpha \in (0,1)\,, \\ & \text{Hess}\left(F^N\right) \text{ is positive definite in } \mathbb{R}^N \setminus \{\mathbf{0}\}\,. \end{aligned} \tag{1.1}$$

In this paper, we are mainly concerned with the physical motivation of studying a certain condenser capacity in an anisotropic environment. More exactly,

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we are dealing with the following free boundary problem:

$$\begin{cases} Qu := \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i}\Big(F^{N-1}(\nabla u)F_{\xi_i}(\nabla u)\Big) = 0 \text{ in } \Omega := \Omega_0 \setminus \overline{\Omega}_1 \subset \mathbb{R}^N, \\ u = 0,\ F(\nabla u) = c_0 \text{ on } \ \partial\Omega_0, \\ u = 1,\ F(\nabla u) = c_1 \ \text{ on } \ \partial\Omega_1. \end{cases} \tag{1.2}$$

Here Ω_0 and Ω_1 are bounded domains of $\mathbb{R}^N$ having boundaries of class C^2, such that $\Omega_0 \supset \overline{\Omega}_1$, while $c_1 > c_0 > 0$ are some real constants. Furthermore, we also assume that Ω_0 and Ω_1 are star shaped with respect to the origin, which is supposed to lie inside Ω_1. By $\nu = (\nu^1, \cdots, \nu^N)$ we denote the outer normal to $\partial\Omega$.

We note that similar problems have been investigated by E. Sartori in [24] and by L. E. Philippin [23] in the case $F(\xi) = |\xi|$ (when Q is the p -Laplace operator, $1 < p < N$), and Laplace operator, respectively. In such cases, problem (1.2) has a weak solution if and only if Ω_0 and Ω_1 are concentric spheres. In our paper, the usual euclidian norm of the gradient is replaced with an arbitrary norm F, satisfying assumption (1.1). The same problem for the case of the anisotropic p -Laplace operator, $1 < p < N$, has already been investigated by L. Barbu-C. Enache in [2], thus the main result of this paper (Theorem 1.1) looks somehow complementary. Studying this class of anisotropic equations could have numerous applications in physics, ranging from some well-established models of surface energies in metallurgy, crystallography, and crystalline fracture theory, to noise-removal procedures in digital image processing (see [3, 4, 5, 6, 7, 8, 9, 20, 21, 22, 26] and references therein).

We will say that $u \in W^{1,N}(\Omega)$ is a weak solution of (1.2) if

$$\int_\Omega F^{N-1}(\nabla u)F_{\xi_i}(\nabla u)v_i \, dx = 0 \quad \text{for any} \quad v \in C_0^\infty(\Omega), \tag{1.3}$$

and $u(\mathbf{x})$ satisfies the boundary conditions $(1.2)_{2,3}$. Regarding the regularity of a solution to problem (1.2), we first note that a solution of the variational problem

$$\operatorname*{Min}_{v\in K} \int_\Omega F^N(\nabla v)\, dx, \tag{1.4}$$

where K is the following set of admissible function

$$K = \left\{ v \in W_0^{1,N}(\Omega_0) \ : \ v \equiv 1 \text{ in } \overline{\Omega}_1 \right\}, \tag{1.5}$$

satisfies

$$Qu = 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega_0, \qquad u = 1 \text{ on } \partial\Omega_1. \tag{1.6}$$

Therefore, we can try to find a solution $u(\mathbf{x})$ of equation $(1.2)_1$ which verifies the first equalities in $(1.2)_{2,3}$, by searching for solutions to variational problem (1.4). According to the regularity theory for quasiminima, the minimizers in (1.5) are bounded, Hölder continuous and satisfy the strong maximum principle (see the book of E. Giusti [13], Theorems 7.5, 7.6 and 7.12). We thus have:

$$0 < u < 1 \text{ in } \Omega. \tag{1.7}$$

Also, since Hess $\left(F^N\right)$ is positive definitive in $\mathbb{R}^N \setminus \{\mathbf{0}\}$, the functional that we minimize in (1.4) is strictly convex, so that the solution $u\left(\mathbf{x}\right)$ is in fact unique. Moreover, since $\partial\Omega \in C^2$, according to G.M. Lieberman [19], P. Tolksdorf [27], we have $u \in C^{1,\alpha}\left(\overline{\Omega}\right)$. Hence boundary conditions $(1.2)_{2,3}$ are well defined. Also, since F verifies assumptions (1.1), equation $(1.2)_1$ is uniformly elliptic in $\Omega \setminus \mathcal{C}$, where $\mathcal{C} := \{\mathbf{x} \in \Omega;\ \nabla u(\mathbf{x}) = \mathbf{0}\}$. Then, the classical regularity theory implies that a weak solution $u \in W^{1,N}(\Omega)$ to equation (1.2) is of class $C^{3,\alpha}$ on $\Omega \setminus \mathcal{C}$ (see O.A. Ladyzhenskaya-N.N. Uraltseva [18]), so that the partial derivatives of $u\left(\mathbf{x}\right)$, up to third order, are well defined on $\Omega \setminus \mathcal{C}$.

Next, let F^* be the *dual norm* of F that is

$$F^*(\mathbf{x}) = \sup_{\xi \neq \mathbf{0}} \frac{\langle \mathbf{x}, \xi \rangle}{F(\xi)} \quad \forall \ \ \mathbf{x} \in \mathbb{R}^N,$$

also called the polar of F. For $r > 0$, we define

$$W_F(r) := \{\mathbf{x} \in \mathbb{R}^N :\ F^*(\mathbf{x}) < r\},\ W_{F^*}(r) := \{\mathbf{x} \in \mathbb{R}^N :\ F(\mathbf{x}) < r\}.$$

In general, for $r > 0$, we say that $W_F(r)$ is the *Wulff shape* (or equilibrium crystal shape) of F, of radius r and center $\mathbf{0}$. A set $D \subset \mathbb{R}^N$ is a Wulff shape of F if there exist $r > 0$ such that $D = \{\mathbf{x} \in \mathbb{R}^N :\ F^\circ(\mathbf{x}) < r\}$. Further details about Wulff shapes may be found in V. Ferone-B. Kawohl [12], A. Cianchi-P. Salani [8].

The main result of this paper states the following:

Theorem 1.1. *If problem* (1.2) *has a weak solution* $u(\mathbf{x})$, *then* $\partial\Omega_1$ *and* $\partial\Omega_0$ *are concentric Wulff shapes, up to translations, whose radii are given by*

$$r_i = \Big(c_i(\ln c_1 - \ln c_0)\Big)^{-1},\ i = 0, 1. \tag{1.8}$$

Moreover, if $F^* \in C^1(\mathbb{R}^N \setminus \{\mathbf{0}\}$, *then the solution* $u(\mathbf{x})$ *is given explicitly by the following formula*

$$u(\mathbf{x}) = \Big((\ln r_0 - \ln r_1)^{-1}(\ln r_0 - \ln F^*(\mathbf{x}))\Big) \ \textit{for any} \quad \mathbf{x} \in \Omega. \tag{1.9}$$

The outline of the paper is as follows. In Section 2 we will prove a maximum principle for an appropriate P-function in the sense of L. E. Payne (see the book of R. Sperb [25]), while in Section 3 this maximum principle will be employed to prove Theorem 1.1.

For convenience, notice that throughout this paper the comma is used to indicate differentiation and the summation from 1 to N is understood on repeated indices. Moreover, we adopt the following notations:

$$\begin{aligned} &F := F(\nabla u), \qquad F_i := F_{\xi_i} = \frac{\partial F}{\partial \xi_i}, \qquad F_\xi = (F_1, ..., F_N), \\ &a_{ij}(\nabla u) := \frac{\partial^2}{\partial \xi_i \partial \xi_j}\left(\frac{1}{p}F^p(\nabla u)\right) = F^{N-1}F_{ij} + (N-1)F^{N-2}F_iF_j, \end{aligned} \tag{1.10}$$

where $i, j \in \{1, ..., N\}$.

2 A maximum principle for an appropriate P-function

In order to prove Theorem 1.1, let us consider the following P-function

$$P(u; \mathbf{x}) := \frac{N-1}{N}F^N(\nabla u(\mathbf{x}))e^{-\alpha u(\mathbf{x})}, \quad \mathbf{x} \in \overline{\Omega}, \tag{2.1}$$

where $u(\mathbf{x})$ is a weak solution to equation $(1.2)_1$, and α is a positive constant chosen to satisfy $P_{|\partial\Omega_0} = P_{|\partial\Omega_1}$. More precisely, α is given by

$$\alpha := N(\ln c_1 - \ln c_0). \tag{2.2}$$

The proof of Theorem 1.1 is presented as a sequence of lemmas. To begin with, we have the following maximum principle:

Lemma 2.1. *Assume that $u(\mathbf{x})$ is a weak solution to problem $(1.2)_1$. Then the auxiliary function P, defined by $(2.1)-(2.2)$, is either identically constant on $\overline{\Omega}$, or it has no interior point of maximum and it satisfies $P_\nu > 0$ on $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1$.*

Here, ν is the exterior unit normal to $\partial\Omega$, while P_ν is the normal derivative of P.

Proof. For the proof of the above maximum principle, the following lemma will play an important role in our computations.

Lemma 2.2. *Assume that* $u(\mathbf{x})$ *is a weak solution to equation* $(1.2)_1$*. Let* a_{ij} *be the coefficients defined by* $(1.10)_2$*. Then the following inequality holds*

$$a_{ij}a_{kl}u_{ik}u_{jl} \geq \frac{(a_{ij}u_{ij})^2}{N} + \frac{N}{N-1}\left[\frac{a_{ij}u_{ij}}{N} - (N-1)F^{N-2}F_iF_ju_{ij}\right]^2 \quad on \ \ \Omega \backslash \mathcal{C}. \tag{2.3}$$

For the proof of Lemma 2.2, we refer the reader to L. Barbu-C. Enache [2], Lemma 2.2.
The proof of Lemma 2.1 is mainly based on the construction of an elliptic differential inequality for the $P(u;\cdot)-$function defined in $(2.1)-(2.2)$ (for computations of this kind see also [1] and [2]).

Since F is positive homogeneous of degree 1, we also have (see G. Wang-C. Xia [28], Proposition 2.1)

$$F_iu_i = F, \qquad F_{ij}u_j = 0, \qquad F_{ijk}u_i = -F_{jk} \qquad \text{for any } \ i \in \{1, ..., N\}\,. \tag{2.4}$$

The following computations are all considered in $\Omega \setminus \mathcal{C}$. We have

$$P_i = e^{-\alpha u}(N-1)\Big(F^{N-1}F_ku_{ki} - \frac{\alpha}{N}F^Nu_i\Big), \tag{2.5}$$

$$\begin{aligned} P_{ij} = \quad & (N-1)e^{-\alpha u}\Big((N-1)F^{N-2}F_lF_ku_{ki}u_{lj} + F^{N-1}F_{kl}u_{lj}u_{ki} \\ & + F^{N-1}F_ku_{kij} - \alpha F^{N-1}F_ku_{ki}u_j \\ & - \alpha F^{N-1}F_ku_{kj}u_i + \tfrac{\alpha^2}{N}F^Nu_iu_j - \tfrac{\alpha}{N}F^Nu_{ij}\Big). \end{aligned} \tag{2.6}$$

Next, making use of notation $(1.10)_2$, we can rewrite $(1.2)_1$ as follows

$$a_{ij}u_{ij} = \big(F^{N-1}F_{ij} + (N-1)F^{N-2}F_iF_j\big)u_{ij} = 0. \tag{2.7}$$

Now, making use of $(2.4), (2.6), (1.10)_2$, after some simplifications, we obtain

$$\begin{aligned} a_{ij}P_{ij} = \quad & (N-1)e^{-\alpha u}\Big((2N-2)F^{2N-3}F_lF_kF_{ij}u_{ki}u_{lj} \\ & + (N-1)^2F^{2N-4}F_lF_kF_iF_ju_{ki}u_{lj} \\ & + F^{2N-2}F_{ij}F_{kl}u_{lj}u_{ki} + F^{N-1}F_ka_{ij}u_{kij} \\ & - 2\alpha(N-1)F^{2N-2}F_kF_iu_{ik} + \alpha^2\tfrac{N-1}{N}F^{2N}\Big). \end{aligned} \tag{2.8}$$

On the other hand, from $(2.4)_1$ and (2.5) one may easily derive the following identities

$$F_ku_{ki} = \frac{P_ie^{\alpha u}}{(N-1)F^{N-1}} + \frac{\alpha}{N}Fu_i \tag{2.9}$$

$$F_iF_ku_{ki} = \frac{\alpha}{N}F^2 + \text{ terms containing } P_m. \tag{2.10}$$

In addition, making use of (2.10) in (2.7), we obtain

$$F_{ij}u_{ij} = -\frac{\alpha(N-1)}{N}F + \text{ terms containing } P_m. \tag{2.11}$$

Differentiating (2.7), we also have

$$\begin{aligned} 0 = \quad & 2(N-1)F^{N-2}F_{il}F_ju_{lk}u_{ij} + (N-1)F^{N-2}F_lF_{ij}u_{lk}u_{ij} \\ & + F^{N-1}F_{ijl}u_{lk}u_{ij} + (N-1)(N-2)F^{N-3}F_iF_lF_ju_{lk}u_{ij} + a_{ij}u_{ijk}. \end{aligned} \tag{2.12}$$

Inserting now $a_{ij}u_{ijk}$ from (2.12) in (2.8), after some simplifications, we derive

$$\begin{aligned} a_{ij}P_{ij} = \quad & (N-1)e^{-\alpha u}\Big((N-1)F^{2N-4}F_lF_kF_iF_ju_{ki}u_{lj} + F^{2N-2}F_{ij}F_{kl}u_{lj}u_{ki} \\ & -2\alpha(N-1)F^{2N-3}F_kF_iF_ju_{ik}u_j + \tfrac{\alpha^2(N-1)}{N}F^{2N} \\ & -(N-1)F^{2N-3}F_kF_lF_{ij}u_{lk}u_{ij} - F^{2N-2}F_kF_{ijl}u_{lk}u_{ij}\Big). \end{aligned} \tag{2.13}$$

Moreover, using (2.9) $-$ (2.11) in (2.13), after some computations, we get

$$\begin{aligned} a_{ij}P_{ij} = \quad & (N-1)e^{-\alpha u}\Big(F^{2N-2}F_{ij}F_{kl}u_{lj}u_{ki} - \tfrac{N-1}{N^2}\alpha^2F^{2N}\Big) \\ & + \text{terms containing } P_m. \end{aligned} \tag{2.14}$$

Next, making use of $(1.10)_2$, (2.9), (2.10), we evaluate separately the term $F^{2N-2}F_{ij}F_{kl}u_{lj}u_{ki}$, as follows:

$$\begin{aligned} & F^{2N-2}F_{ij}F_{kl}u_{lj}u_{ki} = \Big(a_{ij} - (N-1)F^{N-2}F_iF_j\Big) \\ & \times\Big(a_{kl} - (N-1)F^{N-2}F_kF_l\Big)u_{lj}u_{ki} \\ & = a_{ij}a_{kl}u_{lj}u_{ki} + (N-1)^2F^{2N-4}\left(\alpha\frac{F^2}{N} + \text{terms containing } P_m\right)^2 \\ & -2(N-1)F^{2N-4}\Big(FF_{kl} + (N-1)F_kF_l\Big) \\ & \times\left(\frac{\alpha F u_l}{N} + \text{terms containing } P_m\right)\left(\frac{\alpha F u_k}{N} + \text{terms containing } P_m\right) \\ & = a_{ij}a_{kl}u_{lj}u_{ki} - \alpha^2\tfrac{(N-1)^2}{N^2}F^{2N} + \text{terms containing } P_m. \end{aligned} \tag{2.15}$$

Inserting now (2.15) into (2.14), we obtain

$$a_{ij}P_{ij} = (N-1)e^{-\alpha u}\Big(a_{ij}a_{kl}u_{lj}u_{ki} - \alpha^2 F^{2N}\frac{N-1}{N} + \text{terms containing } P_m\Big) . \tag{2.16}$$

Next, to evaluate the term $a_{ij}a_{kl}u_{lj}u_{ki}$ in (2.16), we make use of Lemma 2.2, (2.7) and (2.10). We thus obtain

$$\begin{aligned} a_{ij}a_{kl}u_{lj}u_{ki} &\geq \frac{N}{N-1}\Big((N-1)F^{N-2}F_iF_ju_{ij}\Big)^2 \\ &\geq \frac{N}{N-1}\Big(-\alpha F^N\frac{N-1}{N} + \text{terms containing } P_m\Big)^2 \qquad (2.17) \\ &= \alpha^2 F^{2N}\frac{N-1}{N} + \text{terms containing } P_m . \end{aligned}$$

Therefore, using inequality (2.17) into (2.16), we obtain

$$a_{ij}P_{ij} + \text{terms containing } P_m \geq 0 \text{ in } \Omega \setminus \mathcal{C}. \tag{2.18}$$

Finally, Hopf's first maximum principle (see [16], [25]) implies that P takes its maximum maximum value either on $\partial\Omega$ or at a critical point of $u(\mathbf{x})$. However, P cannot take its maximum at a point where $\nabla u = \mathbf{0}$, since in such a case we would have $P \equiv 0$ on $\overline{\Omega}$, so $u \equiv$ *const.* on $\overline{\Omega}$, which is obviously false. Consequently, either P is identically constant or it attains its maximum over $\overline{\Omega}$ only on $\partial\Omega$, where we then have $P_\nu > 0$, due to Hopf's second maximum principle (see [17], [25]).

The proof of Lemma 2.1 is thus achieved. □

3 The proof of Theorem 1.1

We are going to prove first that if $u(\mathbf{x})$ is a weak solution to problem (1.2), then the auxiliary function P, defined by (2.1)-(2.2), is identically constant on $\overline{\Omega}$. To this end, we will make use of the following two important lemmas.

Next lemma states some properties satisfied by the anisotropic mean curvature of the free boundary.

Lemma 3.1. *If problem* (1.2) *admits a weak solution* $u(\mathbf{x})$*, then the* $F-$ *mean curvature* H_F *of* $\partial\Omega$ *satisfies either*

$$H_{1F} > \alpha\frac{(N-1)c_1}{N} \text{ on } \partial\Omega_1 \qquad \text{and} \qquad H_{0F} < \alpha\frac{(N-1)c_0}{N} \text{ on } \partial\Omega_0, \tag{3.1}$$

or

$$H_{1F} = \alpha\frac{(N-1)c_1}{N} \text{ on } \partial\Omega_1 \qquad \text{and} \qquad H_{0F} = \alpha\frac{(N-1)c_0}{N} \text{ on } \partial\Omega_0, \tag{3.2}$$

where $H_{iF} := H_{F|\partial\Omega_i}\,,\ i = 0, 1.$

Proof. Since $c_i \neq 0,\ i = 0, 1$, equation $(1.2)_1$ is nondegenerate in a neighbourhood of $\partial\Omega$, so it may be rewritten pointwise on $\partial\Omega$ as (see G. Wang-C. Xia [28], Theorem 3.1):

$$0 = Qu = (N-1)F_iF_ju_{ij} - FH_F \quad \text{on} \quad \partial\Omega. \tag{3.3}$$

Therefore, we have

$$H_{0F} = \frac{N-1}{c_0}F_iF_ju_{ij} \quad \text{on} \quad \partial\Omega_0, \quad H_{1F} = \frac{N-1}{c_1}F_iF_ju_{ij} \quad \text{on} \quad \partial\Omega_1. \tag{3.4}$$

Consider now the $P-$ function defined in (2.1) with α as in (2.2). According to Lemma 2.1, two cases may occur. Let us first consider the case when P is not identical constant, so that we have $P_\nu > 0$ on $\partial\Omega$. Since $\nu_F = F_\xi \circ \nu$, and $\langle \nu_F, \nu \rangle = F \circ \nu > 0$ on $\partial\Omega$, ν_F must point outward. From the Dirichlet boundary conditions $(1.2)_{2,3}$, $\nu = -\frac{\nabla u}{|\nabla u|}$ on $\partial\Omega_0$ and $\nu = \frac{\nabla u}{|\nabla u|}$ on $\partial\Omega_1$, thus $\nu_F = -F_\xi(\nabla u)$ on $\partial\Omega_0$ and $\nu_F = F_\xi(\nabla u)$ on $\partial\Omega_1$, therefore

$$\frac{\partial P}{\partial \nu_F} := \langle \nabla P, -F_\xi(\nabla u) \rangle > 0 \text{ on } \partial\Omega_0, \quad \frac{\partial P}{\partial \nu_F} := \langle \nabla P, F_\xi(\nabla u) \rangle > 0 \text{ on } \partial\Omega_1. \tag{3.5}$$

Clearly, the above inequalities yield

$$-\alpha F^2 + NF_iF_ju_{ij} > 0 \text{ on } \partial\Omega_1, \qquad \alpha F^2 - NF_iF_ju_{ij} > 0 \text{ on } \partial\Omega_0, \tag{3.6}$$

where we have used $(2.4)_1$. By combining (3.4) and (3.6) we obtain (3.1). The other case of Lemma 2.1, when P is identically constant on $\overline{\Omega}$, obviously leads to (3.2). The proof is thus achieved. □

Finally, we have

Lemma 3.2. *A necessary condition for the existence of a solution* $u(\mathbf{x})$ *of problem* (1.2) *is*

$$c_0^N \mid \Omega_0 \mid = c_1^N \mid \Omega_1 \mid . \tag{3.7}$$

Proof. Assume first that $u \in C^2(\Omega)$. By divergence theorem we derive

$$
\begin{aligned}
&\int_{\partial\Omega} F^N(\nabla u)\langle \mathbf{x}, \nu\rangle \, d\sigma = \int_{\Omega} \mathrm{div}\, (F^p(\nabla u)\mathbf{x}) \, d\mathbf{x} \\
&\quad = \int_{\Omega} \Big(N F^N(\nabla u) + x_i \big(F^p(\nabla u)\big)_i \Big) \, d\mathbf{x} = N \int_{\Omega} F^N(\nabla u) \, d\mathbf{x} \\
&\quad + N \int_{\Omega} \Big(x_i \big(F^{N-1}(\nabla u) F_k(\nabla u) u_i\big)_k - x_i u_i \big(F^{N-1}(\nabla u) F_k(\nabla u)\big)_k \Big) \, d\mathbf{x} \\
&\quad = N \int_{\Omega} F^N(\nabla u) \, d\mathbf{x} + N \int_{\partial\Omega} \langle \mathbf{x}, \nabla u\rangle F^{N-1}(\nabla u) F_k(\nabla u) \nu_k d\, \sigma \\
&\quad - N \int_{\Omega} F(\nabla u) F_k(\nabla u) u_k \, d\mathbf{x} - N \int_{\Omega} \langle x, \nabla u\rangle \mathrm{div}\, (F^{N-1}(\nabla u) F_\xi(\nabla u)) \, d\mathbf{x} \\
&\quad = N \int_{\partial\Omega} \langle \mathbf{x}, \nabla u\rangle F^{N-1}(\nabla u) F_k(\nabla u) \nu_k d\, \sigma.
\end{aligned}
\tag{3.8}
$$

Since $u = \mathrm{Const.}$ on $\partial\Omega$ we obtain that $u_i = \frac{\partial u}{\partial \nu}\nu_i$, therefore

$$
\int_{\partial\Omega} \langle \mathbf{x}, \nabla u\rangle F^{N-1} F_k \nu_k \, d\sigma = \int_{\partial\Omega} \langle \mathbf{x}, \nu\rangle F^N \, d\sigma. \tag{3.9}
$$

Taking into account $(1.2)_1$ and substituting (3.9) into (3.8) we obtain that

$$
\int_{\partial\Omega} \langle \mathbf{x}, \nu\rangle F^N \, d\sigma = 0. \tag{3.10}
$$

On the other hand

$$
(-1)^i \int_{\partial\Omega_i} \langle \mathbf{x}, \nu\rangle \, d\sigma = N \mid \Omega_i \mid \quad \text{for} \quad i = 0, 1. \tag{3.11}
$$

Using (3.10), (3.11), and $(1.2)_{2,3}$ we derive equality (3.7).

For a weak solution $u(\mathbf{x})$ to problem (1.2), we can use a result obtained by M. Degiovanni, A. Musesti and M. Squassina (see [11], Theorem 2) to conclude that (3.7) holds in fact for $u \in C^{1,\alpha}(\overline{\Omega})$, since F^p is strictly convex. $\square$

Next, we assume contrariwise that P is not identically constant, so that inequalities (3.1) hold. We point out that we have the so-called Minkowski formulas (see, for instance, Y.J. He-H.Li [14], Theorem 1).

$$
(-1)^i \int_{\partial\Omega_i} H_{iF} \cdot \langle \mathbf{x}, \nu\rangle d\sigma = (N-1) \int_{\partial\Omega_i} F \circ \nu \, d\sigma \quad \text{for} \quad i = 0, 1. \tag{3.12}
$$

On the other hand, by the divergence theorem applied to $(1.3)_1$ (working on the approximations and passing to the limit) we have

$$0 = c_0^{N-1} \int_{\partial\Omega_0} F \circ \nu \, d\sigma - c_1^{N-1} \int_{\partial\Omega_1} F \circ \nu \, d\sigma. \tag{3.13}$$

In particular, the starshapedness of Ω_0 and Ω_1 with respect to the origin tell us that $(-1)^i \langle \mathbf{x}, \nu \rangle \geq 0$ on $\partial\Omega_i$ with $(-1)^i \langle \mathbf{x}, \nu \rangle > 0$, $i = 0, 1$, on subsets of positive $(N-1)$ measure. Therefore, multiplying inequalities (3.1) by $\langle \mathbf{x}, \nu \rangle$, integrating over $\partial\Omega_0$, and $\partial\Omega_1$ and using (3.11), we have

$$-\int_{\partial\Omega_1} H_{1F} \langle \mathbf{x}, \nu \rangle \, d\sigma = (N-1) \int_{\partial\Omega_1} F \circ \nu \, d\sigma > (N-1) c_1 \alpha \mid \Omega_1 \mid, \tag{3.14}$$

$$\int_{\partial\Omega_0} H_{0F} \langle \mathbf{x}, \nu \rangle \, d\sigma = (N-1) \int_{\partial\Omega_0} F \circ \nu \, d\sigma < (N-1) c_0 \alpha \mid \Omega_0 \mid. \tag{3.15}$$

Making use of identity (3.13) and inequalities (3.14)-(3.15), we obtain

$$c_1^N \mid \Omega_1 \mid < c_0^N \mid \Omega_0 \mid, \tag{3.16}$$

which contradicts (3.7). Therefore, P is identically constant on $\overline{\Omega}$ and identities (3.2) hold. Consequently, Ω_0 and Ω_1 must be Wulff shapes, whose radii are denoted by r_0, r_1 (see H.J. He-H.Z. Li-H. Ma-J.Q. Ge [15]).

Next, we are going to prove that the solution $u(\mathbf{x})$ to problem (1.2) is given by the formula (1.9), where r_1 and r_2 satisfy (1.8). As we have already mentioned in the Introduction, there exists at most one solution $u(\mathbf{x})$ of equation $(1.2)_1$ which verifies the first equalities in $(1.2)_{2,3}$. Now, we can easily see that the level surfaces of u in Ω are Wulff shapes. Indeed, since the function $P(u; \cdot)$ is constant in Ω, $F(\nabla u)$ is constant on level surfaces of u, therefore we can use the identity obtained in Lemma 3.2 in the set between any level surface and the boundary of Ω_0 or Ω_1 and obtain the claim by using a similar reasoning as above. Therefore, the solution to problem (1.2) is such that all its level sets are Wulff shapes and such that $F(\nabla u)$ is constant on level sets, then it must be of the form $u(\mathbf{x}) = u(F^*(\mathbf{x})) := v(r)$, where $r = F^*(\mathbf{x})$, decreasing as function of r. On the other hand, the solution solution $u(\mathbf{x})$ to equation $(1.2)_1$ which verifies the first equalities in $(1.2)_{2,3}$ can be found by minimizing the functional (1.4). Using the coarea formula and the particular form of u, we can see that the solution to problem (1.2) which verifies the first equalities in $(1.2)_{2,3}$ also minimize the functional

$$\mathcal{J}(v) = \int_{r_0}^{r_1} F^N\big(v'(r) \nabla F^*(x)\big) r^{N-1} dr = \int_{r_0}^{r_1} F^N\big(v'(r)\big) r^{N-1} dr,$$

where we have used the equality $F(\nabla F^*(\mathbf{x})) = 1$ (see Lemma 3.1, [8]). Now, the corresponding Euler-Lagrange equation of this one dimensional problem is the following ordinary differential equation

$$(r^{N-1}(-v'(r))^{N-1})' = 0 \quad \text{on } [r_1, r_0], \tag{3.17}$$

or equivalently

$$rv'(r) = k_0 \quad \text{on } [r_1, r_0], \tag{3.18}$$

where k_0 is a positive constant. Integrating now (3.18), we get

$$u(\mathbf{x}) = k_0 \int_{F^*(\mathbf{x})}^{r_0} s^{-1}\, ds = k_0\Big(\ln r_0 - \ln F^*(\mathbf{x})\Big) \quad \text{on} \quad \Omega. \tag{3.19}$$

Now, writing (3.19) for $F^*(\mathbf{x}) = r_1$, and making use of the boundary condition $(1.2)_3$, we obtain that $k_0 = (\ln r_0 - \ln r_1)^{-1}$, thus we get that the solution u is given explicit by the formula (1.9). Next, differentiating (1.9) with respect to $x_i,\ i \in \{1, \cdots, n\}$, we obtain that

$$\nabla u(\mathbf{x}) = -\frac{\nabla F^*(\mathbf{x})}{(\ln r_0 - \ln r_1)F^*(\mathbf{x})}. \tag{3.20}$$

Finally, in order to derive (1.8) we use the boundary conditions $(1.2)_{2,3}$, (3.20) and the identity $F(\nabla F^*(\mathbf{x})) = 1$.

This achieves the proof of Theorem 1.1.

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